Symbolic extensions

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1 Introduction

The theory of symbolic extensions is concerned the following two questions regarding a topological dynamical system (X, T):

QUESTION 1:
Does there exist a topological symbolic extension (Y, σ) of (X, T)? In other words, is (X, T) a topological factor of some subshift?

QUESTION 2:
If yes, what is the infimum of the topological entropies of all its symbolic extensions?

It turns out, that in order to be able to answer these questions, one needs to focus not only on the topological entropies of the involved systems (the system (X, T) and its symbolic extensions (Y, σ)), but also on the measure-theoretic (Kolmogorov-Sinai) entropies of all invariant measures supported by these systems. The two key notions of the theory are defined below.

Definition 1.1. *Let (X, T) be a topological dynamical system. The topological symbolic extension entropy of (X, T) is defined as follows:*

$\mathbf{h}_{\mathsf{sex}}(X, T) = \inf\{\mathbf{h}_{\mathsf{top}}(Y, \sigma) : (Y, \sigma)$ *is a subshift extension of* $(X, T)\}$;
$\mathbf{h}_{\mathsf{sex}}(X, T) = \infty$ *if* (X, T) *has no symbolic extensions.*

A refinement of this notion at the level of invariant measures is this:

Definition 1.2. *Let (X, T) be a topological dynamical system and let $\mathcal{P}_T(X)$ denote the set of all T-invariant measures μ on X. Let (Y, S) be a topological extension of (X, T) and $\pi : Y \to X$ be the corresponding factor map. On $\mathcal{P}_T(X)$ we define the extension entropy function by the formula*

$$h^\pi(\mu) = \sup\{h_\nu(S) : \nu \in \mathcal{P}_S(Y), \pi^*(\nu) = \mu\}.$$

Then, on $\mathcal{P}_T(X)$ we define the symbolic extension entropy function, by

$h_{\mathsf{sex}}(\mu) = \inf\{h^\pi(\mu) : \pi : Y \to X$ *is a subshift extension of* $(X, T)\}$;
$h_{\mathsf{sex}}(\mu) = \infty$ *(for all measures μ) if (X, T) has no symbolic extensions.*

One of the fundamental tools in the theory of symbolic extensions is the following theorem (one inequality is obvious, the other requires some machinery):

Theorem 1.3 (Symbolic Extension Entropy Variational Principle)**.**

$$\mathbf{h}_{\mathsf{sex}}(X,T) = \sup\{h_{\mathsf{sex}}(\mu) : \mu \in \mathcal{P}_T(X)\}.$$

The main task of the theory of symbolic extensions reduces to solving the following problem:

PROBLEM 1:
Compute (or estimate) h_{sex} *for a given system* (X,T) *using its internal properties.*

Notice that the definition of h_{sex} is so constructed, that solving Problem 1 answers both of the formerly formulated questions 1 and 2. In full generality, so phrased problem has been solved in the paper [3], and then refined in [12]. The solution is in terms of so-called *entropy structure*, a carefully selected sequence of functions on $\mathcal{P}_T(X)$, which reflects the emergence of the entropy of different measures at refining scales. Crucial are upper semicontinuity properties of these functions and the multiple defect of uniformity in its convergence. The reason why these items are so essential can very roughly and briefly explained as follows: In the system (X,T) some invariant measures may reveal all of its entropy already in large scale (like in expansive systems), other measures may need very small scale (i.e., fine covers) for their entropy to be detected. Now, in the symbolic extension (Y,σ), the small scale dynamics must be "magnified" and become visible in the large scale of the symbolic system (in symbolic system all dynamics happens in large scale). If the "large scale measures" are approximated in $\mathcal{P}_T(X)$ by the "small scale measures", the magnification of small scale dynamics may lead to enlarging the entropy of large scale dynamics. This causes the overall entropy of the symbolic extension to grow.

We will also address a more particular problem, concerning smooth maps:

PROBLEM 2:
Estimate h_{sex} *or* $\mathbf{h}_{\mathsf{sex}}$ *for* C^r*-maps on manifolds* $(1 \leqslant r \leqslant \infty)$*.*

We will show how this problem is solved in dimension one, i.e., for smooth maps of the interval or of the circle, in terms of much more familiar parameters, such as the degree of smoothness r and the (slightly refined) Lipshitz constant.

2 The history of research on topological symbolic extensions

The first result concerning symbolic extensions in topological dynamics is due to William Reddy and goes back to 1968 [22]. It says that every expansive homeomorphism T on a compact metric space has a symbolic extension. The construction provided no control over the entropy of this extension.

It was clear that expansiveness was a much too strong requirement. All known examples of finite entropy systems seemed to admit symbolic extensions. One of the spectacular applications of symbolic extensions occurs in the studies of hyperbolic systems. Using Markov partitions, such systems can be lifted to subshifts of finite type, which allows to apply symbolic dynamical methods to the hyperbolic systems. This approach belongs to the classics, it is described for example in Bowen's book [1]. Generally, however, very little was known. The natural question whether all finite entropy systems indeed have symbolic extensions has been presumably puzzling many people between the years 1970 and 1990. Around 1989, Joe Auslander addressed this question to Mike Boyle, one of the best experts in symbolic dynamics. Within some time, Boyle came up with the negative answer, by constructing an appropriate example. A version of the same example showed, that even if a system does admit a symbolic extension, there may exist a necessary gap between the entropy of the system and that of any symbolic extension. He called this gap *the residual entropy.* These examples have been presented at the Adler conference in 1991, but never published until 2002. They proved only one thing: there is no easy answer to the questions 1 and 2 stated in the Introduction.

For the next 8 years, the progress was rather limited and not published. Mike Boyle collaborated in this matter with Doris and Ulf Fiebig. They tried to construct symbolic extensions by means of symbolic and topological methods (without using invariant measures), which, from today's perspective, explains why their results were so restricted.

Around 1998 the same problem was ancountered by the author of this note. Together with Fabien Durand, we were characterizing all factors of so-called Toeplitz flows, and one of the three conditions for a system to be such a factor was that it admits some symbolic extension ([13]). It soon occurred, that nobody knew any general criteria for that. Mike Boyle was able to say that any system of entropy zero has a symbolic extension also of entropy zero ([2]), which was very useful for the study of factors of Toeplitz flows.

In year 1999, I spent a month in Marseille, devoting all my energy trying to understand why some systems have and other do not have symbolic extensions. For simplicity, I focused on zero dimensional systems, which seemed to be the best class to study. I have discovered that the existence of symbolic extensions depends on the distribution of entropy on invariant measures, which lead to the first result containing the criteria for the existence and an estimate of the topological entropy of symbolic extensions for general zero-dimensional systems ([11]). In particular, it was shown that an asymptotically h-expansive zero-dimensional system admits a symbolic extension of the same topological entropy.

A year later, Boyle and the Fiebigs publish an extensive paper containing the results of their long lasting collaboration ([4]). The old examples appear here in the original version, next to new ones, where the transformation is on a disc and is differentiable at all but one point. In terms of positive results, all asymptotically h-expansive systems (not necessarily zero-dimensional) are shown to possess *principal* symbolic extensions, i.e., such that not only the topological entropy is the same as that of (X,T), but also the Kolmogorov-Sinai entropy of every invariant measure is the same as that of its image in the system (X,T). Since expansive systems are asymptotically h-expansive, we recover here a refined version of Reddy's first result. Since any system of entropy zero is asymptotically h-expansive, we also recover the fact communicated earlier by Boyle to the author of this note. Another spectacular application, neatly included in [4] concerns smooth maps. Soon before that, Jerome Buzzi just proved that any C^∞ map on a Riemannian manifold is in fact asymptotically h-expansive ([9]). (Many years earlier Sheldon Newhouse proved a seemingly weaker statement [21], which from today's perspective is equivalent to Buzzi's result.) Now, this fact receives a new meaning: every C^∞ map on a manifold admits a principal symbolic extension. If we agree, that symbolic extensions are "lossless digitalizations", then principal symbolic extensions can be regarded "gainless" (without superfluous information) digitalizations. The fact that all C^∞ maps can be losslessly and gainlessly digitalized became one of the iconic achievements of the theory of symbolic extensions. However, an immediate question arises: what about C^r maps, where $r < \infty$?

In 2001, the author of this note visited Mike Boyle. Leaving the smooth systems aside, we worked on the general theory. Our article [3] contains the complete and general characterization of the symbolic extension entropy function h_{sex}. It also contains the aforementioned variational principle for the symbolic extension entropy. Problem 1 and both questions stated in the preceding section, became completely solved. The solution still refers to zero-dimensional systems: each system with finite entropy is first shown to posses a principal zero-dimensional extension (using the theory of *mean dimension*, by E. Lindenstrauss and B. Weiss [18], [17]) and then it is shown how to build a symbolic extension of a zero-dimensional system. The notion of an *entropy structure* is introduced for zero-dimensional systems, the key tool to compute the symbolic extension entropy function. A criterion is provided, when the symbolic extension entropy function is *attained*, i.e., when a symbolic extension exist, whose entropy function matches the symbolic extension entropy function (an "optimal" digitalization).

Next year, the author of this work develops a consistent theory of entropy structures for general topological dynamical systems ([12]). Among other things, this allows to simplify the phrasing of several results from the preceding work, by skipping the intermediate stage of a zero-dimensional extension. The theory of entropy structures, although its importance depends upon the application to symbolic extensions, has gained an independent interest and several papers appeared devoted to other aspects of the entropy structure theory ([8], [19]).

At the same time I collaborated with Sheldon Newhouse. The focus of this collaboration was on smooth maps on Riemannian manifolds. The obtained results ([16]) are of negative nature: roughly speaking they prove that (in some class) a typical C^1 system of dimension $d \geqslant 2$ admits no symbolic extensions at all (infinite symbolic extension entropy), while a typical C^r map, where $1 < r < \infty$ (also for $d \geqslant 2$) does not admit a principal symbolic extension (without deciding whether it does admit a symbolic extension). In the examples the gap between the entropy of the system and the entropy of a symbolic extension (the residual entropy) is bounded below by some term (which we denote here by **R**) proportional to the logarithm of the Lipshitz constant and inverse proportional to $r-1$. We formulated a conjecture, that the residual entropy in our examples is the worst, i.e., that every C^r map with $r>1$ does admit a symbolic extension, and the symbolic extension entropy is, in the worst case, equal to the entropy plus **R**.

This conjecture triggered a number of papers containing partial results. In all cases the conjecture has been confirmed: In 2005, jointly with Alajandro Maass, we have proved the conjecture true in dimension $d=1$ ([15]). The breakthrough in proving this occurred in quite extraordinary circumstances – during my trip to Antarctica. This is why we have decided to call the key lemma in the proof the *Antarctic Theorem*. This result was then complemented by David Burguet, who provided examples of C^r interval maps showing the estimate **R** for the residual entropy to be sharp ([5]). Recently, in a series of papers Burguet proved the conjecture for arbitrary C^r maps on surfaces ([6, 7]). The general case of a C^r map (or diffeomorphism) on a compact manifold of dimension d remains an open problem, and the latest Burguet's result for $d=2$ is the most advanced step toward the full solution.

The recent book [10] offers a unified approach to the theory of entropy with focus on symbolic extensions and entropy structures. Many former proofs are simplified and the notation is made more consistent.

3 Theory of superenvelopes

This and the following sections provide some details of the theory of symbolic extensions. The goal is to give formulations of the main results. We have reduced the amount of definitions and auxiliary facts to the minimum necessary for an understandable presentation of the theorems.

Exceptionally, throughout this section the letter X will denote an abstract compact domain. In the applications to entropy structures it will be replaced by the set of invariant measures $\mathcal{P}_T(X)$, rather than the phase space X of the dynamical system. Similarly, the points x of the domain will be replaced by invariant measures denoted most often by μ.

Let us remind that a function $f : X \to \mathbb{R}$ is *upper semicontinuous*, if every set of the form $\{x \in X : f(x) < t\}$ is open. Every upper semicontinuous function defined on a compact set is bounded from above. The class of upper semicontinuous functions i sclosed under finite sums and arbitrary infima.

Let $\mathcal{F} = (f_k)$ denote a nondecreasing sequence of nonnegative functions on X, such that $f_0 \equiv 0$ and the differences $f_k - f_{k-1}$ are upper semicontinuous ($k \geqslant 1$). In particular, all functions f_k are upper semicontinuous (but this is not an equivalent condition). We will also assume that all these functions are commonly bounded, i.e., that the limit function $f(x) = \lim_k f_k(x)$ is bounded. For the purposes of this note such a sequence will be called a *structure*.

Definition 3.1. *By a* superenvelope *of a structure $\mathcal{F}$ we will understand any* *function E on X satisfying*

- *$E \geqslant f_k$ for every k (equivalently, $E \geqslant f$),*
- *the difference $E - f_k$ is upper semicontinuous for every k.*

Additionally we will also admit the constant infinity function as a superenvelope of any structure.

In particular we obtain that if E is a finite superenvelope then $E - f$ is upper semicontinuous but this is not an equivalent condition. It is easy to see that the family of all superenvelopes (which is nonempty) is closed under taking arbitrary infima. Thus there exists a *minimal superenvelopea* of the structure $\mathcal{F}$. It will be denoted by $\mathsf{E}\mathcal{F}$.

In the general case the superenvelopes of a structure may behave very strangely. Some structures have no finite seperenvelopes (ans then $E\mathcal{F} \equiv \infty$). In every other case it holds that $\mathsf{E}\mathcal{F}$ is a nonnegative upper semicontinuous function, and thus it is bounded from above. Moreover, it equals the limit function f on a residual set (i.e., on a dense set of type G_δ). However, the pointwise suprema of f and of $E\mathcal{F}$ may differ significantly.

We will now descibe a method of determining $\mathsf{E}\mathcal{F}$ using transfinite inductoin. Let us remind that for a given function $g : X \to \mathbb{R}$ its *upper semicontinuous envelope* denoted by $\widetilde{g}$ is defined as the smallest upper semicontinuous function greater than or equal to g. Again, applying the convention that the infimum of an empty family is infinite, we must agree that $\widetilde{g} \equiv \infty$ for any g unbounded from above.

Let θ_k denote the difference $f - f_k$ (in general this is not an upper semicontinuous function). Next, for the countable ordinals $\alpha < \omega_1$ we define the transfinite sequence u_α (depending on the structure $\mathcal{F}$) as follows:

- $u_0 \equiv 0$
- once we have defined u_β for all $\beta < \alpha$ we first set
$$v_\alpha = \sup_{\beta<\alpha} u_\beta,$$
and then we define
$$u_\alpha = \lim_k \{\widetilde{v_\alpha + \theta_k}\}$$

(clearly, this is a decreasing limit, bounded from below). It is not hard to prove (using compacness of the domain) that the transfinite sequence u_α grows only up to some countable ordinal α_0, above which it becomes constant, i.e., $u_\alpha \equiv u_{\alpha_0}$ for all $\alpha > \alpha_0$. The smallest such index α_0 is called *the order of accumulation of the structure $\mathcal{F}$*. Three important cases are possible: Either (1) the function v_α is unbounded for some α. Then the smallest such α becomes our order of accumulation α_0 and all functions u_α for $\alpha \geqslant \alpha_0$ are constant infinity, or (2) all the functions v_α are bounded and then the sequence u_α is (commonly) bounded, hence u_{α_0} is bounded. Now we can provide the formula for the smallest superenvelope:

Theorem 3.2.

$$E\mathcal{F} = f + u_{\alpha_0}.$$

We can see that the case (1) is equivalent to the condition $\mathsf{E}\mathcal{F} \equiv \infty$ (there does not exist a finite superenvelope), while the case (2) corresponds to the existence of (at least one) finite superenvelope (for instance $\mathsf{E}\mathcal{F}$) which is then automatically bounded.

We will distinguish one more important parameter, namely the number

$$c^* = \sup\{u_1(x) : x \in X\}.$$

It is easy to prove that the following conditions are equivalent:

1. $c^* = 0$,
2. $u_1 \equiv 0$,
3. $u_{\alpha_0} \equiv 0$,
4. $E\mathcal{F} \equiv f$,
5. the structure (f_n) converges uniformly to the limit function f.

Such very special structures will play an important role in the theory, in particular in the terminal section of this note.

We have to enrich our consideration by assuming that the dmain X is a convex subset of some Banach space. It will be so in the application to dynamical systems, where X will be replaced by $\mathcal{P}_T(X)$.

In this case we will be interested in *affine structures* $\mathcal{F}$, i.e., such all the functions f_k are *affine* (i.e., preserve convex combinations). For such structures, important for us will be the *affine superenvelopes*, i.e., simply, the superenvelopes E_{A} which are affine functions. The family of such superenvelopes is no longer closed under infima, hence the (in general) there is no such thing as the minimal affine superenvelope. Nonetheless, the following theorem holds:

Theorem 3.3. *If $\mathcal{F}$ is an affine structure on a convex domain X, then the pointwise infimum of all affine superenvelopes equals the minimal superenvelope $\mathsf{E}\mathcal{F}$. That is to say, for every $x \in X$ we have,*

$$\inf\{E_{\mathsf{A}}(x) : E_{\mathsf{A}} \textit{ is an affine superenvelope}\} = \mathsf{E}\mathcal{F}(x).$$

In particular, this implies that $\mathsf{E}\mathcal{F}$ is, for such a structure, a concave function.

4 Enrtopy structures and the symbolic extension entropy theorem

In this section we give the definition of an entropy structure as a carefully selected affine structure defined on the compact convex set $\mathcal{P}_T(X)$ of all invariant measures in a topological dynamical system (X, T), and we provide a formula for the symbolic extension entropy in terms of superenvelopes.

Definition 4.1. *A Borel partition $\mathcal{P}$ of the space X is said to have* small boundary *if the union of the boundaries of all elements of $\mathcal{P}$ has measure zero for any measure $\mu \in \mathcal{P}_T(X)$.*

Alas, not every dynamical system has enough small boundary partitions to generate the Borel sigma-algebra, in some easy examples there are no even non-trivial sets with small boundaries. However, introducing small modifications in our system (inessential for the existence and entropy of symbolic extensions), the existence of such partitions can be obtained. We will thus assume, without loss of generality that there exists a sequence of partitions $(\mathcal{P}_k)$ with the following properties for every $k \geqslant 1$:

1. The partition $\mathcal{P}_k$ has small boundaries;
2. The partition $\mathcal{P}_{k+1}$ is a refinement of $\mathcal{P}_k$;
3. Maximal diameter of an atom of the partition $\mathcal{P}_k$ does not exceed 2^{-k}.

Once such a sequence of partitions is given, we can define the key notion in the theory, so-called entropy structure:

Definition 4.2. *Let (X, T) be a topological dynamical system with finite topological entropy. The* entropy structure *of this system is the sequence $\mathcal{H} = (h_k)$, where $h_k : \mathcal{P}_T(X) \to [0, \infty)$ is given by*

$$h_k(\mu) = h_\mu(\mathcal{P}_k, T),$$

where $(\mathcal{P}_k)$ is a sequence of partitions satisfying the above conditions 1.–3.

(In case the system does not have such a sequence of partitions the entropy structure is defined in an alternative, more general, way, however the details of such a definition are a bit too complicated for this survey; see [12]. The structure then has the same properties as described below for the structure defined using the partitons with small boundaries.)

The general well-known properties of the measure-theoretic entropy imply that the functions h_k are nonnegative, affine on $\mathcal{P}_T(X)$ and commonly bounded by the number $\mathbf{h}_{\mathsf{top}}(T)$. As a consequence of the assumptions made on the the sequence of partitions $(\mathcal{P}_k)$ we have further, from our point of view crucial, properties of the entropy structure: the sequence $\mathcal{H} = (h_k)$ is nondecreasing, its limit equals the entropy function h (given by $h(\mu) = h_\mu(T)$), and, most importantly, for every $k \geqslant 1$ the difference $h_{k+1} - h_k$ is upper semicontinuous (we additionally define $h_0 \equiv 0$). So, $\mathcal{H}$ is an affine structure defined on $\mathcal{P}_T(X)$ and all facts stated in the preceding section apply. In particular, we know that $\mathsf{E}\mathcal{H}$ equals the pointwise infimum of all affine superenvelopes E_{A}.

We are in a position to formulate the main theorem of the theory of symbolic extensions.

Theorem 4.3 (Symbolic extension entropy theorem)**.** *Let (X, T) be a topological dynamical system with finite entropy and let $\mathcal{H}$ denote its entropy structure. Let E be a real-valued function defined on $\mathcal{P}_T(X)$. The following conditions are equivalent:*

- $E = E_{\mathsf{A}}$ *is an affine superenvelope of the entropy structure* $\mathcal{H}$*;*
- *There exists a symbolic extension* (Y, σ) *of* (X, T) *(via a factor map* $\pi : Y \to X$*) such that on* $\mathcal{P}_T(X)$ *the function* E *equals* h^π*.*

As immediate corollaries we obtain that

- $h_{\mathsf{sex}}(\mu) = \mathsf{E}\mathcal{H}(\mu)$, for any $\mu \in \mathcal{P}_T(X)$,
- $\mathbf{h}_{\mathsf{sex}}(X, T) = \sup\{\mathsf{E}\mathcal{H}(\mu) : \mu \in \mathcal{P}_T(X)\}$,
- (X, T) possesses a symbolic extension if and only if there exists a finite superenvelope of the entropy structure $\mathcal{H}$ (we need not to seek an affine superenvelope here).

In this manner we have succeeded in determining the parameters decisive for the possibility of creating a symbolic extension of a system. It is important that we establish these parameters based exclusively on internal properties of the system, that is *without* needing to examine all its symbolic extensions. In particular, we are able to decide whether the system has a symbolic extension and (for example) how large an alphabet is needed for it, *before* we even attempt to build one.

Entropy structure allows to characterize another important class, so-called asymptotically h-expansive systems. The notion has been introduced by M. Misiurewicz together with an entropy-like parameter denoted by $\mathbf{h}^*(T)$, whose original definition we will skip (see [20]). Asymptotic h-expansiveness is defined as the condition $\mathbf{h}^*(T) = 0$. Although in general there is no immediate inequality between topological entropy $\mathbf{h}_{\mathsf{top}}(T)$ and the Misiurewicz parameter $\mathbf{h}^*(T)$, nonetheless the following implication holds:

$$\mathbf{h}_{\mathsf{top}}(T) = 0 \implies \mathbf{h}^*(T) = 0$$

(i.e., systems with topological entropy zero are asymptotically h-expansive). It turns out that the Misiurewicz parameter (and hence asymptotic h-expansiveness) depend directly on the entropy structure and its parameters, as follows:

Theorem 4.4. *For a topological dynamical system* (X, T) *the equality holds:*

$$\mathbf{h}^*(T) = c^*,$$

where c^* *is evaluated (as the pointwise supremum of the function* u_1*) for the entropy structure* $\mathcal{H}$*.*

This implies further characterizations of asymptotically h-expansive systems, this time expressed in terms of the entropy structure and symbolic extensions.

Theorem 4.5. *The following conditions are equivalent:*

1. *The system* (X, T) *is asymptotically* h*-expansive;*

2. *The entropy structure converges* uniformly *to the entropy function;*

3. $h_{\mathsf{sex}}(\mu) = h_\mu(T)$ *for every invariant measure* μ *on* X*;*

4. *The system* (X,T) *possesses a* principal *symbolic extension, i.e., such that* $h^\pi \equiv 0$.

5 Symbolic extensions of smooth interval maps

A spectacular example of an important class of dynamical systems for which the theory of entropy structures allows to very precisely establish the value of the topological symbolic extension entropy (without needing to build such an extension) is the class of smooth interval maps. These are fundamental objects in the theory of dynamical system and hence they are well studied and fairly well understood. Nonetheless, the existence of symbolic extension was, until recently, completely undecided.

Let $I = [0,1]$ be our model interval. We now say what we understand by the "degree of smoothness" of a map $T : I \to I$.

Definition 5.1. *If* $r \in (0,1]$*, we will say that* T of class C^r *if* T *is* Hölder continuous *with parameter* r*, i.e,. for any* $x, y \in I$*, satisfies*

$$|T(x) - T(y)| \leqslant |x - y|^r.$$

For $r > 1$ *we require inductively that* T *be differentiable and its derivative* T' *be of class* C^{r-1}.

It is not hard to see that $r_1 < r_2$ implies $C^{r_1} \supset C^{r_2}$, that any function differentiable n times with continuous nth derivative is of class C^n, and that the class C^∞ consists of functions differentiable infinitely many times.

For a function T differentiable with continuous derivarive we also define some other parameters. The first of them is

$$L(T) = \sup\{\log |T'(x)| : x \in I\}.$$

This number coincides with the logarithm of the *Lipshitz constant* for T. Next we define

$$R(T) = \max\left\{0, \lim_n \tfrac{1}{n} L(T^n)\right\},$$

where T^n denotes the nth iterate (not the usual power) of T, $T^n = T \circ T \circ \cdots \circ T$ (n times). We will call this parameter the *expansive constant*. Further, for an ergodic measure μ on I the number

$$\chi_0(\mu) = \max\left\{0, \int \log |T'(x)|\, d\mu(x)\right\}$$

is called the *Lyapunov exponent* of μ. Both the expansive constant and the Lyapunov exponents are fundamental notions in the ergodic theory of smooth systems. The basic inequalities regarding the entropy of the interval maps are:

- For any ergodic measure μ we have $h_\mu(T) \leqslant \chi_0(\mu)$;
- $\mathbf{h}_{\mathsf{top}}(T) \leqslant R(T)$.

(Both are consequences of the so-called *Margulis–Ruelle inequality*, see [23]).

The theory of entropy structures allows to estimate, in a similar manner, the symbolic extension entropy function, which is, in a sense, a culmination point of this theory.

Theorem 5.2 (The Antarctic Theorem)**.** *Let T be an interval map of class C^r with $r > 1$. Then:*

- *For any ergodic measure μ we have $h_{\mathsf{sex}}(\mu) \leqslant h_\mu(T) + \frac{\chi_0(\mu)}{r-1}$;*
- $\mathbf{h}_{\mathsf{sex}}(T) \leqslant \mathbf{h}_{\mathsf{top}}(T) + \frac{R(T)}{r-1}$.

In the proof one shows a much more specific fact. Namely, one proves that the function $\frac{\chi_0(\mu)}{r-1}$ (extended affinely to all of $\mathcal{P}_T(I)$) is an affine superenvelope of the entropy structure, which means that there exists a symbolic extension (Y, σ) of (I, T) (via some factor map $\pi : Y \to I$) such that for ergodic measures μ we have the equality

$$h^\pi(\mu) = \frac{\chi_0(\mu)}{r-1}.$$

In particular, the theorem provides an answer to one of the basic questions one might ask about symbolic extensions for intervals maps:

- Every C^r interval map (with $r > 1$) admits a symbolic extension.

One of important corollaries of the above theorem is that if T is of class C^∞ then it possesses a principal symbolic extension i.e., such that $h^\pi \equiv 0$. That means that the system (I, T) can be digitalized not only losslessly but also "gainlessly", i.e., without superfluous information contents. Another corollary is that systems of class C^∞ are asymptotically h-expansive, which is a fact of independent interest (although does not refer directly to symbolic extensions). These facts hold not only for interval maps but for C^∞ transformations of any compact Riemannian manifolds and were proved in this wider generality even before the interval case ([9], [4]).

Let us also notice that for $r = 1$ the estimates in the Antarctic Theorem return infinite values. And indeed, there are examples of C^1 interval maps which have no symbolic extensions (see [5]).

Further details of the theory of symbolic extensions and entropy structures exceed the scope of this article. The reader will find them in the original papers and in the book [10].

Based on a lecture delivered at
University of Chile
Santiago, 2008

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